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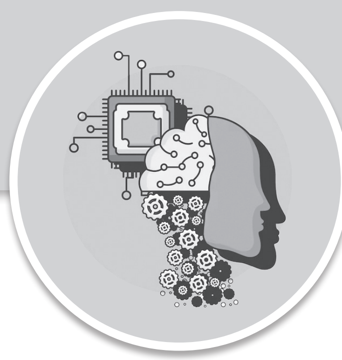
Linear Algebra



Comprehensive Theory
with Solved Examples *and* Practice Questions

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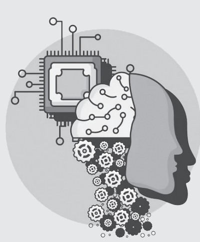

EDITIONS

First Edition : 2025

Second Edition : 2026

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1

Vector Space and Subspace

1.1 VECTOR SPACE

A vector space (or linear space) over a field F is a set V equipped with two operations:

1. **Vector Addition:** A function $+ : V \times V \to V$, which maps each pair of vectors $u, v \in V$ to a vector $u + v \in V$.
2. **Scalar Multiplication:** A function $\cdot : F \times V \to V$, which maps each scalar $a \in F$ and vector $v \in V$ to a vector $a \cdot v \in V$.

These operations must satisfy the following axioms for all $u, v, w \in V$ and all scalars $a, b \in F$:

1.1.1 Vector Addition

For every pair of vectors $u, v \in V$, there exists a unique vector $u + v \in V$, such that the following properties hold:

1. **Commutative Law:**
$$u + v = v + u, \ \forall u, v \in V$$
2. **Associative Law:**
$$(u + v) + w = u + (v + w), \forall u, v, w \in V$$
3. **Existence of Additive Identity (Zero Vector):** There exists a unique element $0 \in V$ such that
$$v + 0 = v, \forall v \in V$$
4. **Existence of Additive Inverse:** For every $v \in V$, there exists a unique vector $-v \in V$ such that
$$v + (-v) = 0$$

1.1.2 Scalar Multiplication

For every scalar $a, b \in F$ and every vector $u, v \in V$, the following properties hold:

1. **Closure under Scalar Multiplication:**
$a.v \in V, \forall a \in F, v \in V$
2. **Distributive Law (Scalar over Vector Addition):**
$$a \cdot (u + v) = a \cdot u + a \cdot v, \forall a, \in F, u, v \in V$$
3. **Distributive Law (Field Addition over Scalar Multiplication):**
$$(a + b) \cdot v = a \cdot \mathrm{v} + b \cdot v, \forall a, b \in F, v \in V$$
4. **Associativity of Scalar Multiplication:**
$$(a \cdot b) \cdot v = a \cdot (b \cdot v), \forall a, b \in F, v \in V$$
5. **Existence of Multiplicative Identity:** There exists a unique scalar $1 \in F$ such that
$$v = v, \forall v \in V$$

1.2 SUBSPACE

Let $(V, +, .)$ be a vector space over a field F. A subset $U \subseteq V$ is called a **subspace** of V if the following conditions hold:

1. **Non-emptiness (Zero Vector):** The zero vector 0 of V must be in U. This ensures that U is non-empty and contains the neutral element of the vector space.
2. **Closure under addition:** For all $u_1, u_2 \in U$, the sum $u_1 + u_2$ must also belong to U. That is, U is closed under vector addition..
3. **Closure under scalar multiplication:** For any scalar $a \in F$ and any vector $u \in U$, the scalar multiple $a.u$ must also belong to U. Thus, U is closed under scalar multiplication.

1.2.1 Properties of Subspaces

- **Containment of Zero Vector:** Every subspace contains the zero vector. This is a consequence of closure under scalar multiplication since $a \cdot u = 0$ when $a = 0$.
- **Linearity:** Since subspaces inherit the vector space structure, they satisfy all the axioms of a vector space, including associativity, commutativity of addition, existence of additive inverses, and distributivity of scalar multiplication.
- **Non-empty.** A subspace is always non-empty because it must contain the zero vector.

Example:

1. **The Zero Subspace:** The set {0}, consisting only of the zero vector, is a subspace of any vector space. It satisfies all conditions because
   - The zero vector is in the set.
   - The sum of two zero vectors is the zero vector.
   - Any scalar multiple of the zero vector is the zero vector.
2. **The Entire Space:** The vector space V itself is trivially a subspace of V because it contains the zero vector, is closed under addition and scalar multiplication, and satisfies all the axioms of a vector space.
3. **Lines through the Origin:** A line through the origin in a vector space V is a subspace. For instance the line spanned by a non-zero vector v is given by:

   $$L = \{\alpha \cdot v \mid \alpha \in F\}$$

   This set is closed under addition and scalar multiplication, contains the zero vector ($0.v = 0$), and is a subspace.
4. **Planes through the Origin:** Similarly, in R^3, the plane containing the origin, spanned by two non-parallel vectors, is a subspace it satisfies closure under addition and scalar multiplication, and contains the zero vector.

1.2.2 Important Theorems Related to Subspaces

- **The Subspace Test:** A subset U of a vector space V is a subspace if and only if it satisfies the following three conditions:
  1. $0 \in U$ (the zero vector is in U).
  2. U is closed under addition.
  3. U is closed under scalar multiplication.
- **Intersection of Subspaces:** The intersection of two subspaces U_1 and U_2 of V is also a subspace of V. This is because the intersection is closed under addition and scalar multiplication, and it contains the zero vector.

Example 1.1 Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Span $\mathbb{R}^2$.

Solution:

We need to check if we can express any vector (x, y) as a linear combination.

$$c_1 v_1 + c_2 v_2 = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Expanding:

$$\begin{bmatrix} c_1 + 2c_2 \\ 3c_1 + 6c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Writing as an augmented matrix:

$$\text{Perform } R_2 \leftarrow R_2 - 3R_1 \begin{bmatrix} 1 & 2 & x \\ 3 & 6 & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & x \\ 0 & 0 & y - 3x \end{bmatrix}$$

The second row is $0 = y - 3x$, which means the system is **inconsistent for some x, y.** Since the second row is zero, the two vectors are **linearly dependent** (one is a multiple of the other). This means they do not span $\mathbb{R}^2$.

So, No, v_1 and v_2 do not span $\mathbb{R}^2$ because they are dependent.

Example 1.2 Do the vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Span $\mathbb{R}^2$?

Solution:

We check if any (x, y) can be written as:

$$c_1 v_1 + c_2 v_2 = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 - c_2 \\ c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad 2c_1 - c_2 = x \quad \text{...(i)}$$

$$\Rightarrow \quad c_1 + 3c_2 = y \quad \text{...(ii)}$$

Writing as an augmented matrix:

$$\left[\begin{array}{cc|c} 2 & -1 & x \\ 1 & 3 & y \end{array}\right]$$

Student's Assignments

Q.1 Consider the vectors:

$u = [1, -2, 3]^T$, $v = [-2, 4, -6]^T$ in R^3

Which of the following set can be used to uniquely represent both vectors?

(a) $\{[1, 0, 0]^T, [0, 0, 1]^T\}$
(b) $\{[1, -2, 3]^T\}$
(c) $\{[1, -2, 3]^T, [2, 4, -6]^T\}$
(d) None of these

Q.2 If u and v are orthogonal vector in the inner product space v, such that $\|u\| = 6$ and $\|u + v\| = 0$ then what is the value of $\|v\|$ and $\|u - v\|$ is ________.

(a) 4, 2 (b) 8, 12
(c) 6, 12 (d) $8, \sqrt{12}$

Q.3 Let V_1 and V_2 be a subspace of a vector space V. Which of the following is necessarily a subspace of V?

(a) $V_1 \cup V_2$
(b) $V_1 \cap V_2$
(c) $\frac{V_i}{V_2} = \{x \in v_1 \text{ and } y \notin v_2\}$
(d) $V_1 + V_2 = \{x + y \mid x \in v_1{}' y \in V_2\}$

Q.4 Let V_1 and V_2 be subspace of 4 dimensional vector space V, both having dimension 3. What is the smallest possible dimension of their intersection $V_1 \cap V_2$?

(a) 1 (b) 2
(c) 4 (d) 3

Q.5 Let S and T be non-empty subsets and subspaces of $\mathbb{R}^2$, and W be a non-zero proper subspace of $\mathbb{R}^2$. Consider the following statement. Find which one is not correct:

(a) $S \subseteq T$ then Span(S) $\subseteq$ (T)
(b) Span($S \cup T$) = Span(S) + Span(T)
(c) Span($S \cap T$) $\subseteq$ Span(S) $\cap$ Span(T)
(d) If Span(S) = $\mathbb{R}^2$, then Span($S \cap W$) = W

Q.6 Which of the following is a subspace of the real vector space $\mathbb{R}^3$?

(a) $\{(x, y, z) \in \mathbb{R}^3 : (3y + 2z)^2 + (2x - 3y)^2 = 0$
(b) $\{(x, y, z) \in \mathbb{R}^3 : y \in \mathbb{Q}\}$
(c) $\{(x, y, z) \in \mathbb{R}^3 : xy = 0)$
(d) $\{(x, y, z) \in \mathbb{R}^3 : 5x + 7y - 3z + 1 = 0\}$

Q.7 Let $\{e_1, e_2, e_3\}$ be a basis of a vector space V over $\mathbb{R}$. Consider the following sets:

$A = \{e_2, e_1 + e_2, e_1 + e_2 + e_3\}$
$B = \{e_1 e_1 + e_2, e_1 + e_2 + e_3\}$
$C = \{2e_1, 3e_1 + e_3, 6e_1 + 3e_2 + e_3\}$

(a) A and B are bases of V
(b) A and C are bases of V
(c) B and C are bases of V
(d) Only B is a basis of V

Q.8 U is a subset of $\mathbb{R}^4$ given by $\{(x_1 - x_2 + x_3 = 0 = x_1 + x_2 + x_4)\}$ then

(a) U is not a subspace of $\mathbb{R}^4$
(b) U is a subspace of $\mathbb{R}^4$ of dimension 1
(c) U is a subspace of $\mathbb{R}^4$ of dimension 2
(d) U is a subspace of $\mathbb{R}^4$ of dimension 3

Q.9 Let $S = \{x_1, x_2, ..., x_m\}$ and $T = \{y_1, ..., y_n\}$ be subsets of the vector space V. Then

(a) If S and T are both linearly independent then $m = n$
(b) If S is a basis for V and if T spans V then $m \geq n$
(c) If S is a basis for V and if T is linearly independent, then $m \geq n$
(d) If S is linearly independent and if T spans V, then $m \leq n$

Q.10 Let S be the set of all $n \times n$ matrices over R with zero trace. Then

(a) S is not a vector space
(b) S is a vector space of dimension $n - 1$
(c) S, together with the identity matrix, form a vector space
(d) S is a vector space and it has a basis consisting of $n^2 - 1$ matrices

Q.11 Let W be the subspace of $\mathbb{R}^2$ spanned by (1, 2). Which of the following pairs represent the same element of the quotient space $\frac{\mathbb{R}^2}{W}$.

(a) $W, \left(\frac{1}{2}, 3\right)+W$

(b) $\left(1, \frac{1}{2}\right)+W, \left(6, \frac{21}{2}\right)+W$

(c) $\left(1, \frac{1}{2}\right)+W, \left(6, \frac{3}{2}\right)+W$

(d) $\left(\sqrt{2}, 1\right)+W, \left(\sqrt{2}, 2\right)+W$

Q.12 Suppose V_1, V_2, V_3, V_4 are linearly independent vectors of a real vector space. Consider the two sets of vectors

$S_1 = \{V_1 + V_2, V_1 + V_3, V_1 + V_4\}$

$S_2 = \{V_1 + V_2, V_1 + V_3, V_1 + V_4, V_4 + V_1\}$

Which of the following is true?

(a) Both S_1 and S_2 are linearly independent.

(b) S_2 is linearly independent but not S_1.

(c) S_1 is linearly independent but not S_2.

(d) Neither S_1 nor S_2 is linearly independent.

Answers Keys

1. (b)	**2.** (c)	**3.** (b, d)	**4.** (b)	**5.** (d)
6. (a)	**7.** (a, b, c)	**8.** (c)	**9.** (c, d)	**10.** (d)
11. (b)	**12.** (c)			

Explanations

1. (b)

$$A = \{[1, -2, 3]^T\}$$

$$u = [1, -2, 3]^T, v = [-2, 4, -6]^T$$

Since u is already in A, It is trivial represent as:

$$u = 1 \cdot [1, -2, 3]^T$$

If there exists a scalar c such that

$$v = c \cdot [1, -2, 3]^T$$

Substituting $V = [-2, 4, -6]^T$

$$[-2, 4, -6]^T = c \cdot [1, -2, 3]^T$$

Solving for c

$$c \cdot 1 = -2$$

$$\Rightarrow \quad c = -2$$

$$c \cdot (-2) = 4$$

$$\Rightarrow \quad c = -2$$

$$c \cdot 3 = -6$$

$$\Rightarrow \quad c = -2$$

Since $c = -2$ satisfied all three equations.

$$v = -2 \cdot [1, -2, 3]^T$$

Since, both u and v can be uniquely written as scalar multiples of the single vector in A the set $\{[1, -2, 3]^T\}$.

2. (c)

The norm of vector is zero. If and only if the vector itself is the zero vector.

$$\|u+v\| = 0$$

$$u + v = 0$$

$$v = -u$$

$$\|v\| = \|-u\| = \|u\| = 6$$

So, $V = -u$

$$\|u-v\| = \|u-(-u)\| = \|u+u\| = \|2u\|$$

$$= 2 \times 6 = 12$$

4. (b)

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

$\dim(V_1 + V_2) = 3 + 3 - \dim(V_1 \cap V_2)$

Since $V_1 + V_2$ is a subspace of V.

It's dimension cannot exceed 4.

i.e., $\dim(V_1 + V_2) \le 4$

Smallest Dimension of $V_1 \cap V_2$

$$4 \ge 6 - \dim(V_1 \cap V_2)$$

$$\dim(V_1 \cap V_2) \ge 2$$

So, smallest possible dimension of

$$V_1 \cap V_2 = 2$$

5. (d)

Let $S = \{(1, 0), (0, 1)\} \Rightarrow \text{Span}(S) = \mathbb{R}^2$

Let $W = \text{Span}\{(1, 1)\} \Rightarrow$ a proper subspace (a line)

Then $S \cap W = \phi \Rightarrow \text{span}(S \cap W) = \{0\} \subseteq W$

So, even if span $(S) = \mathbb{R}^2$, Span $(S \cap W) \neq W$ in general.

So option (d) is not always true.